

POISSON SUPERALGEBRAS AS NONASSOCIATIVE ALGEBRAS

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ABSTRACT. Poisson superalgebras are known as a \mathbb{Z}_2 -graded vector space with two operations, an associative supercommutative multiplication and a super bracket tied up by the super Leibniz relation. We show that we can consider a single nonassociative multiplication containing all these datas and then consider Poisson superalgebra as non associative algebras.

1. GENERALITIES ON POISSON ALGEBRAS

1.1. Poisson algebras. Let \mathbb{K} be a field of characteristic 0. A \mathbb{K} -Poisson algebra is a \mathbb{K} -vector space \mathcal{P} equipped with two bilinear products denoted by $x \cdot y$ and $\{x, y\}$, having the following properties:

- (1) The couple (\mathcal{P}, \cdot) is an associative commutative \mathbb{K} -algebra.
- (2) The couple $(\mathcal{P}, \{, \})$ is a \mathbb{K} -Lie algebra.
- (3) The products \cdot and $\{, \}$ satisfy the Leibniz rule:

$$\{x \cdot y, z\} = x \cdot \{y, z\} + \{x, z\} \cdot y,$$

for any $x, y, z \in \mathcal{P}$.

The product $\{, \}$ is usually called Poisson bracket and the Leibniz identity means that the Poisson bracket acts as a derivation of the associative product.

In [3], one proves that any Poisson structure on a \mathbb{K} -vector space is also given by a nonassociative product, denoted by xy and satisfying the non associative identity

$$(1) \quad 3A(x, y, z) = (xz)y + (yz)x - (yx)z - (zx)y.$$

where $A(x, y, z)$ is the associator $A(x, y, z) = (xy)z - x(yz)$. In fact, if \mathcal{P} is a Poisson algebra given by the associative product $x \cdot y$ and the Poisson bracket $\{x, y\}$, then xy is given by

$$xy = \{x, y\} + x \cdot y.$$

Conversely, the Poisson bracket and the associative product of \mathcal{P} are the skew-symmetric part and the symmetric part of the product xy . Thus it is equivalent to present a Poisson algebra classically or by this nonassociative product. In [1], we have studied algebraic properties of the nonassociative algebra \mathcal{P} . In particular we have proved that this algebra is flexible, power-associative and admits a Pierce decomposition.

If \mathcal{P} is a Poisson algebra given by the nonassociative product (1), we denote by $\mathfrak{g}_{\mathcal{P}}$ the Lie algebra on the same vector space \mathcal{P} whose Lie bracket is

$$\{x, y\} = \frac{xy - yx}{2}$$

and by $\mathcal{A}_{\mathcal{P}}$ the commutative associative algebra, on the same vector space, whose product is

$$x \cdot y = \frac{xy + yx}{2}.$$

An important problem in mathematical physics and more precisely in Quantum Field theory is the deformation of Poisson algebras. The classical deformations of Poisson algebras consist of deformations of the Poisson brackets, that is, deformations of $\mathfrak{g}_{\mathcal{P}}$ which let the associative multiplication of $\mathcal{A}_{\mathcal{P}}$ unchanged and satisfying the Leibniz identity [4]. In [5] this type of deformation has been generalized by using a nonassociative multiplication defining the Poisson structure. The deformations of this nonassociative multiplication provides general Poisson deformations.

2. POISSON SUPERALGEBRA

By a \mathbb{K} -super vector space, we mean a \mathbb{Z}_2 -graded vector space $V = V_0 \oplus V_1$. The vectors of V_0 and V_1 are called homogeneous vectors of degree respectively equal to 0 and 1. For an homogeneous vector x , we denote by $|x|$ its degree. A \mathbb{K} -Poisson superalgebra is a \mathbb{K} -super vector space $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1$ equipped with two bilinear products denoted by $x \cdot y$ and $\{x, y\}$, having the following properties:

- The couple (\mathcal{P}, \cdot) is a associative super commutative \mathbb{K} -algebra, that is,

$$x \cdot y = (-1)^{|x||y|} y \cdot x.$$

- The couple $(\mathcal{P}, \{, \})$ is a \mathbb{K} -Lie super algebra, that is,

$$\{x, y\} = -(-1)^{|x||y|} \{y, x\}$$

and satisfying the super Jacobi condition:

$$(-1)^{|z||x|} \{x, \{y, z\}\} + (-1)^{|x||y|} \{y, \{z, x\}\} + (-1)^{|y||z|} \{z, \{x, y\}\} = 0.$$

- The products \cdot and $\{, \}$ satisfy the super Leibniz rule:

$$\{x, y \cdot z\} = \{x, y\} \cdot z + (-1)^{|x||y|} y \cdot \{x, z\}.$$

where x, y and z are homogeneous vectors.

Theorem 1. *Let \mathcal{P} a \mathbb{K} -super vector space. Thus \mathcal{P} is a Poisson superalgebra if and only if there exists on \mathcal{P} a nonassociative product xy satisfying*

$$(2) \quad \begin{cases} 3(xy)z - 3x(yz) + (-1)^{|x||y|}(yx)z - (-1)^{|y||z|}(xz)y - (-1)^{|x||y|+|x||z|}(yz)x \\ + (-1)^{|x||z|+|y||z|}(zx)y = 0 \end{cases}$$

for any homogeneous vectors $x, y, z \in \mathcal{P}$.

Proof. Assume that $(\mathcal{P}, \cdot, \{, \})$ is a Poisson superalgebra. Consider the multiplication

$$xy = x \cdot y + \{x, y\}.$$

We deduce that

$$x \cdot y = \frac{1}{2}(xy + (-1)^{|x||y|}yx).$$

Thus the associativity condition writes for homogeneous vectors

$$\begin{aligned} v_1(x, y, z) &= A(x, y, z) - (-1)^{|x||y|+|x||z|+|y||z|}A(z, y, x) + (-1)^{|x||y|}(yx)z \\ &\quad - (-1)^{|y||z|}x(zy) - (-1)^{|x||y|+|x||z|}(yz)x + (-1)^{|x||z|+|y||z|}z(xy) \\ &= 0 \end{aligned}$$

where $A(x, y, z) = (xy)z - x(yz)$. Likewise, the Poisson bracket writes for homogeneous vectors

$$\{x, y\} = \frac{1}{2}(xy - (-1)^{|x||y|}yx)$$

and the super Jacobi condition

$$\begin{aligned} v_2(x, y, z) &= (-1)^{|x||z|}A(x, y, z) - (-1)^{|x||y|+|x||z|}A(y, x, z) - (-1)^{|x||y|+|y||z|}A(z, y, x) \\ &\quad - (-1)^{|x||z|+|y||z|}A(x, z, y) + (-1)^{|x||y|}A(y, z, x) + (-1)^{|y||z|}A(z, x, y) \\ &= 0 \end{aligned}$$

The super Leibniz writes

$$\begin{aligned} v_3(x, y, z) &= A(x, y, z) - (-1)^{|x||y|}A(y, x, z) + (-1)^{|x||y|+|x||z|+|y||z|}A(z, y, x) \\ &\quad + (-1)^{|y||z|}A(x, z, y) + (-1)^{|x||y|+|x||z|}A(y, z, x) - (-1)^{|x||z|+|y||z|}A(z, x, y) \\ &= 0. \end{aligned}$$

Let us consider the vector

$$\begin{aligned} v(x, y, z) &= \frac{1}{3} [(-1)^{|x||y|}(yx)z - (-1)^{|y||z|}(xz)y - (-1)^{|x|(|y|+|z|)}(yz)x + (-1)^{(|x|+|y|)|z|}(zx)y] \\ &\quad + (xy)z - x(yz). \end{aligned}$$

Then

$$v(x, y, z) = \frac{1}{6} (2v_1(x, y, z) + (-1)^{|x||z|}v_2(x, y, z) + v_3(x, y, z) + 2(-1)^{|x||z|+|y||z|}v_3(z, x, y)).$$

We deduce that the product xy satisfies

$$v(x, y, z) = 0$$

for any homogeneous vectors x, y, z .

Conversely, assume that the product of the non associative product \mathcal{P} satisfies $v(x, y, z) = 0$ for any homogeneous vectors x, y, z . Let $v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)$ be the vectors of \mathcal{P} defined in the first part respectively in relation with the associativity, the super Jacobi and super Leibniz relations. We have

$$\begin{aligned} v_1(x, y, z) &= v(x, y, z) - (-1)^{|x||y|+|x||z|+|y||z|}v(z, y, x) + (-1)^{|y||z|}v(x, z, y) \\ &\quad - (-1)^{|x||z|+|y||z|}v(z, x, y) \\ v_2(x, y, z) &= (-1)^{|x||z|}v(x, y, z) - (-1)^{|x||y|+|x||z|}v(y, x, z) - (-1)^{|x||y|+|y||z|}v(z, y, x) \\ &\quad - (-1)^{|x||z|+|y||z|}v(x, z, y) + (-1)^{|x||y|}v(y, z, x) + (-1)^{|y||z|}v(z, x, y) \\ v_3(x, y, z) &= v(x, y, z) - (-1)^{|x||y|}v(y, x, z) + (-1)^{|x||y|+|x||z|+|y||z|}v(z, y, x) \\ &\quad + (-1)^{|y||z|}v(x, z, y) + (-1)^{|x||y|+|x||z|}v(y, z, x) - (-1)^{|x||z|+|y||z|}v(z, x, y) \end{aligned}$$

Examples. Any 2-dimensional superalgebra $\mathcal{P} = V_0 \oplus V_1$ with an homogeneous basis $\{e_0, e_1\}$ is defined

$$\begin{cases} e_0 e_0 = a e_0, \\ e_0 e_1 = b e_1, \quad e_1 e_0 = c e_1, \\ e_1 e_1 = d e_0. \end{cases}$$

This is a super Poisson multiplication if and only if we have

$$\begin{cases} d = 0, \\ 3(a - b)b + ab - 2bc + c^2 = 0, \\ 3(a - c)c + ab - 2bc + c^2 = 0, \end{cases}$$

or

$$\begin{cases} a = 0, \\ a = b = c. \end{cases}$$

We obtain the following 2-dimensional Poisson superalgebras

$$\begin{cases} \mathcal{SP}_{2,1} & e_0 e_0 = a e_1 \\ \mathcal{SP}_{2,2} & e_0 e_0 = a e_1, \quad e_0 e_1 = e_1 e_0 = a e_1 \\ \mathcal{SP}_{2,3} & e_0 e_1 = -e_1 e_0 = b e_1 \\ \mathcal{SP}_{2,4} & e_1 e_1 = d e_0, \end{cases}$$

the non written product being considered equal to 0. Let us note that these 2-dimensional algebras correspond to the algebras $(\mu_{16}, \beta_2 = 0)$, $(\mu_{16}, \beta_2 = 1)$, $(\mu_9, \alpha_2 = 0, \beta_4 = 0)$, $(\mu_5, \alpha_2 = 0)$ in the classification [2].

3. PROPERTIES OF POISSON SUPERALGEBRAS

Definition 2. A nonassociative superalgebra is called *super flexive* if the multiplication xy satisfy

$$A(x, y, z) + (-1)^{(|x||z|+|x||y|+|y||z|)} A(z, y, x) = 0$$

for any homogeneous elements x, y, z , where $A(x, y, z) = (xy)z - x(yz)$ is the associator of the multiplication.

Proposition 3. Let \mathcal{P} be a Poisson superalgebra. Then the non associative product defining the super Poisson structure is super flexive.

Proof. In fact, let

$$B(x, y, z) = 3(A(x, y, z) + (-1)^{(|x||z|+|x||y|+|y||z|)} A(z, y, x)).$$

We have

$$\begin{aligned} B(x, y, z) &= -(-1)^{|x||y|}(yx)z + (-1)^{|y||z|}(xz)y + (-1)^{|x||y|+|x||z|}(yz)x - (-1)^{|x||z|+|y||z|}(zx)y \\ &\quad + (-1)^{(|x||z|+|x||y|+|y||z|)}(-(-1)^{|z||y|}(yz)x + (-1)^{|y||x|}(zx)y \\ &\quad + (-1)^{|z||y|+|z||x|}(yx)z - (-1)^{|z||x|+|y||x|}(xz)y) \\ &= (-(-1)^{|x||y|} + (-1)^{|x||y|})(yx)z + ((-1)^{|y||z|} - (-1)^{|y||z|})(xz)y \\ &\quad + ((-1)^{|x||y|+|x||z|} - (-1)^{|x||y|+|x||z|})(yz)x + (-(-1)^{|x||z|+|y||z|} \\ &\quad + (-1)^{|x||z|+|y||z|})(zx)y \\ &= 0. \end{aligned}$$

Remark : On the power associativity. Recall that a nonassociative algebra is power associative if every element generates an associative subalgebra. Let \mathcal{P} be a Poisson superalgebra provided with its non associative product xy . If V_0 is its the even homogeneous part, then the restriction of the product xy is a multiplication in this homogeneous vector space satisfying Identity (2). Since all the vectors of V_0 are of degree 0, Identity (2) is reduced to Identity (1). We deduce that V_0 is a Poisson algebra and any vector x in V_0 generates an associative subalgebra of V_0 and of \mathcal{P} .

Assume now that y is an odd vector. We have

$$y \cdot y = \frac{1}{2}(yy + (-1)yy) = 0,$$

and

$$\{y, y\} = \frac{1}{2}(yy - (-1)yy) = yy.$$

If we write $y^2 = yy$, then

$$y^2 = \{y, y\}.$$

This implies

$$yy^2 = y\{y, y\} = y \cdot \{y, y\} + \{y, \{y, y\}\}.$$

But from the super identity of Jacobi, $\{y, \{y, y\}\} = 0$. Thus we have

$$yy^2 = y \cdot \{y, y\} = \{y, y\} \cdot y = y^2y.$$

We can write

$$y^3 = yy^2 = y^2y.$$

Now

$$y^2y^2 = \{y, y\}\{y, y\} = \{y, y\} \cdot \{y, y\} + \{\{y, y\}, \{y, y\}\}.$$

We have also

$$yy^3 = y \cdot y^3 + \{y, y^3\} = y \cdot y \cdot \{y, y\} + \{y, y \cdot \{y, y\}\}.$$

But $y \cdot y = 0$. Thus, from the Leibniz rule,

$$yy^3 = \{y, y \cdot \{y, y\}\} = -y \cdot \{y, \{y, y\}\} + \{y, y\} \cdot \{y, y\} = \{y, y\} \cdot \{y, y\}.$$

We deduce

$$y^2y^2 - yy^3 = \{\{y, y\}, \{y, y\}\}.$$

Since $\{y, y\}$ is of degree 0, we obtain

$$y^2y^2 - yy^3 = 0.$$

We can write

$$y^4 = y^2y^2 = yy^3 = y^3y$$

the last equality results of $\{y, y \cdot \{y, y\}\} = \{y \cdot \{y, y\}, y\}$. Now, using Identity (2) to the triple (y^i, y^j, y^k) with $i + j + k = 5$, we obtain a linear system on the vectors $y^i y^j$ with $i + j = 5$, which admits as solutions

$$yy^4 = y^2y^3 = y^3y^2 = yy.$$

Thus y^5 is well determinated. By induction, using Identity (2) on the triple (y^i, y^j, y^k) with $i + j + k = n$, using induction hypothesis $y^p y^{n-1-p} = y^{n-1}$, we obtain that

$$y^n = y^p y^{n-p}$$

for any $p = 1, \dots, n-1$. Thus any homogeneous element of odd degree generates an associative algebra.

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